October Math Gems

PROBLEM OF THE WEEK 7

§1 Problems

Problem 1.1. The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a+b+c$.

Solution. $\boxed{\textbf{Answer: 98}}$
We have that $9x^3 = (x+1)^3$, so it follows that $\sqrt[3]{9}x = x+1$. Solving for x yields $\frac{1}{\sqrt[3]{9}-1} = \frac{\sqrt[3]{81}+\sqrt[3]{9}+1}{8}$, so the answer is $\boxed{98}$. $\square$

Problem 1.2. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a+b+c$.

Solution. $\boxed{\textbf{Answer: 263}}$
So we add 4 to both sides:

$$1 + \frac{3}{x-3} + 1 + \frac{5}{x-5} + 1 + \frac{17}{x-17} + 1 + \frac{19}{x-19} = x^2 - 11x$$

And simplify:

$$\frac{x}{x-3} + \frac{x}{x-5} + \frac{x}{x-17} + \frac{x}{x-19} = x^2 - 11x$$

Divide out by x (giving us 0 as a solution for x):

$$\frac{1}{x-3} + \frac{1}{x-5} + \frac{1}{x-17} + \frac{1}{x-19} = x - 11$$

Now we notice that this is symmetric about $x - 11$, so substitute $x = 11 + y$:

$$\frac{1}{y-8} + \frac{1}{y-6} + \frac{1}{y+6} + \frac{1}{y+8} = y$$

Now we can put the fractions under common denominators:

$$\frac{(y+8)+(y-8)}{(y+8)(y-8)} + \frac{(y+6)+(y-6)}{(y+6)(y-6)} = y$$

And simplify:

$$\frac{2y}{y^2-64} + \frac{2y}{y^2-36} = y$$

Divide out by y (giving us 11 as a solution for x):

$$\frac{2}{y^2-64}+\frac{2}{y^2-36}=1$$

We multiply out by $(y^2-64)(y^2-36)$ and expand:

$$2y^2-128+2y^2-72=y^4-100y^2+2304$$

We bring everything to one side:

$$y^4-104y^2+2504=0$$

This is a quadratic in y^2, so we solve:

$$\begin{aligned} y^2 &= \frac{104\pm\sqrt{104^2-4\cdot 2504}}{2} \\ &= \frac{104\pm\sqrt{800}}{2} \\ &= \frac{104\pm 2\sqrt{200}}{2} \\ &= 52\pm\sqrt{200} \end{aligned}$$

So $y=\pm\sqrt{52\pm\sqrt{200}}$ and $x=11\pm\sqrt{52\pm\sqrt{200}}$. So our solutions for x are $11\pm\sqrt{52\pm\sqrt{200}}$. Obviously the maximum of these is $11+\sqrt{52+\sqrt{200}}$, so our answer is $11+52+200=\boxed{263}$. □

Problem 1.3. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b\geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x)=56$$

$$\log_{\log(x)}(x)=54$$

and finds that this system of equations has a single real number solution $x>1$. Find b.

Solution. **Answer: 216**

The idea is that

$$\log_{\log(x)} x=\frac{\log x}{\log(\log x)}$$

by change of base. Now set $m=\log x, n=\log(\log x)$. Because

$$\frac{3}{2}\log x+3\log(\log x)=56$$

after splitting the logarithm, we have $\frac{3}{2}\cdot 54b+3b=56 \implies 84b=56, b=\frac{2}{3}$. Therefore $\log x=36$, and

$$\log_b(36)=\frac{2}{3},$$

which means $b=\boxed{216}$. □

Problem 1.4. Let r, s, and t be the three roots of the equation

$$8x^3+1001x+2008=0.$$

Find $(r+s)^3+(s+t)^3+(t+r)^3$.

Solution. **Answer: 753**

Theorem (Vieta's Formula)
Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients and degree n, having complex roots $r_n, r_n - 1, \ldots, r_1$. Then for any integer $0 \le k \le n$

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{1i} r_{2i} \ldots r_{ik} = (-1)^k \frac{a_{n-k}}{a_n}$$

By Vietas, we know that our answer should be $-(r^3 + s^3 + t^3)$.
Now, we also know, by Vietes, $r + s + t = 0$
We have:

$$8r^3 + 1001r + 2008 = 0$$

$$8s^3 + 1001s + 2008 = 0$$

$$8t^3 + 1001t + 2008 = 0$$

Adding these, we get
$8(r^3 + s^3 + t^3) = -2008 \cdot 3$
So our answer is $\boxed{753}$.

□

Problem 1.5. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

Solution. **Answer: 2**

$$x^2 - 10x = -y^2 - 6x - 34$$

Completing the square,

$$x^2 - 10x + 25 = -y^2 - 6x - 9$$

Factoring,

$$(x-5)^2 = (-y-3)(y+3)$$

When $x = 5$, $y = -3$, so $5 + (-3) = 2$

□

Problem 1.6. What is the value of

$$\frac{2^{2014} + 2^{2012}}{2^{2014} - 2^{2012}}?$$

Solution. $\boxed{\textbf{Answer: } \frac{5}{3}}$
Let $2^{2012} = x$. We now have $\frac{4x+x}{4x-x}$, which is equal to $\frac{5}{3}$. □

Problem 1.7. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

Solution. $\boxed{\textbf{Answer: } -7007}$
By polynomial division we have that

$$f(x) = g(x)(x + \frac{c}{10}) \leftrightarrow x^4 + x^3 + bx^2 + 100x + c = (x^3 + ax^2 + x + 10)(x + \frac{c}{10})$$

Now equating coefficients,

$$\frac{c}{10} + a = 1, \frac{ac}{10} + 1 = b, \frac{c}{10} + 10 = 100$$

This means that $c = 900, a = -89$, $b = -90 \cdot 89 + 1$. So, the answer is

$$900 + 100 + 2 + 1 - 90 \cdot 89 = 1003 - 8010 = -7007$$

□

Problem 1.8. Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

Solution. $\boxed{\textbf{Answer: } 881}$
In order to simplify the problem, we turn the outside logarithms into their exponent form (multiple times):

$$\log_{2^a}(\log_{2^b}(2^{1000})) = 1$$
$$\log_{2^b}(2^{1000}) = 2^a$$
$$(2^b)^{2^a} = 2^{1000}$$

Simplifying this equation gives us that $b \cdot 2^a = 1000$. Since $1000 = 2^3 \cdot 5^3$, the possibilities for (a, b) are $(1, 500), (2, 250)$, and $(3, 125)$. Therefore, the sum of all possible values of $a + b$ is $501 + 252 + 128 = \boxed{881}$. □

Problem 1.9. A hexagon that is inscribed in a circle has side lengths 22, 22, 20, 22, 22, and 20 in that order. The radius of the circle can be written as $p + \sqrt{q}$, where p and q are positive integers. Find $p + q$.

Solution. $\boxed{\textbf{Answer: } 272}$
Let the vertices of the hexagon be A, B, C, D, E, F, in that order, such that $AB = 20$ and $DE = 20$. Let the center of the circle be O. Observe that $\triangle OAB \cong \triangle OED$ and that all the remaining triangles with one side 22 formed by connecting O to the vertices of the hexagon are congruent. Thus letting the central angle $\angle AOB = y$ and the central angles of the other four congruent triangles be x, we have $2x + y = 180$, which means points C, O, F are collinear. This in turn implies that $ABCF$ is an inscribed isosceles trapezoid. Dropping an altitude from A to $\overline{CF}$, we use the pythagorean theorem to get

$$r^2 - 10^2 = 22^2 - (r - 10)^2$$

and solving the resulting quadratic in r for its positive real value gives $r = 5 + \sqrt{267}$ $\implies p + q = \boxed{272}$. □

Problem 1.10. What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

Solution. $\boxed{\textbf{Answer: } 10}$
Note that $x^2 - 3x + 2 = (x-1)(x-2)$ has roots 1 and 2. By vieta's formulas, the sum of the roots of $x^2 - 5x + k$ is 5.

So $x^2 - 5x + k$ can have roots 1 and 4, or 2 and 3. Using vieta's formulas again, k is the product of the roots.

Thus, our answer if $1 \cdot 4 + 2 \cdot 3 = 4 + 6 = \boxed{10}$. $\square$

Problem 1.11. If $y + 4 = (x-2)^2, x + 4 = (y-2)^2$, and $x \neq y$, what is the value of $x^2 + y^2$?

Solution. $\boxed{\textbf{Answer: } 15}$
If we subtract the four from our starting equations, we have that

$$y = x^2 - 4x$$
$$x = y^2 - 4y$$

If we add these two equations, then we have that $x^2 + y^2 = 5(x + y)$. Put that aside for now.

Substituting $x = y^2 - 4y$ into $y = x^2 - 4x$ and simplifying, we get that

$$y(y^3 - 8y^2 + 12y + 15) = 0$$

Testing values using the Rational Root Theorem, we see that $y - 5$ is a factor of the cubic, so

$$y(y-5)(y^2 - 3y - 3) = 0$$

If $y = 0$, then $4 = (x-2)^2$. Then $x = 0$ or $x = 4$. However, $x \neq y$ and $x = 4 \Rightarrow 8 = (-2)^2$.

If $y = 5$, then $9 = (x-2)^2$. Then $x = 5$ or $x = -1$. However, again $x \neq y$ and $x = -1 \Rightarrow 3 = 3^2$.

Thus, y is a root of $y^2 - 3y - 3$. Since our system of equations is reflexive and $x \neq y$, by similar casework x must be the root of $x^2 - 3x - 3$ that y isn't. Then by Vieta's, $x + y = 3$. Thus $x^2 + y^2 = 5(x + y) = 5 \cdot 3 = \boxed{15}$. $\square$

Problem 1.12. Points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ are distinct points on the graph of $y^2 + x^4 = 2x^2y + 1$. What is $|a - b|$?

Solution. $\boxed{\textbf{Answer: } 2}$
We rearrange the equation to:

$$x^4 - 2x^2y + y^2 = 1$$

Notice that $x^4 - 2x^2y + y^2 = (x^2 - y)^2$.

$$(x^2 - y)^2 = 1$$

So, we replace the coordinates.

$$(\pi - b) = 1$$

$$(\pi - a) = -1$$

$$a = \pi - 1$$

$$b = \pi + 1$$

So our answer is $1 + 1 = \boxed{2}$.

Problem 1.13. Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Solution. $\boxed{\textbf{Answer: } 469}$

We can see that whenever we have an exponent x^{n^2}, we can make it into $(x^n)^n$.

So that means $a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2} = 27^{\log_3 7} + 49^{\log_7 11} + (\sqrt{11})^{\log_{11} 25}$.

Note that $27 = 3^3$, $49 = 7^2$, and $\sqrt{11} = 11^{\frac{1}{2}}$.

Then we can get $3^{3\log_3 7} + 7^{2\log_7 11} + 11^{\frac{1}{2}\log_{11} 25} = 3^{\log_3 7^3} + 7^{\log_7 11^2} + 11^{\log_{11} 25^{\frac{1}{2}}}$.

This is simplified to (with laws of exponents) $7^3 + 11^2 + 25^{\frac{1}{2}} = 343 + 121 + 5 = \boxed{469}$. □

Problem 1.14. How many positive integers n satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

Solution. $\boxed{\textbf{Answer: } 6}$

Take k to be an integer such that $k \le \sqrt{n} < k + 1 \Rightarrow k^2 \le n < k^2 + 2k + 1$. The given equality rearranges to $n = 70k - 1000$. The first inequality $k^2 \le 70k - 1000$ gives $20 \le k \le 50$. The second inequality $70k - 1000 < k^2 + 2k + 1$ simply rearranges to $155 < (k - 34)^2$. Combining these 2 equations, we see that $k = 20, 21, 47, 48, 49, 50$. It is not hard to see that each k gives a unique valid n, so the answer is $\boxed{6}$ □

Problem 1.15. In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of $BFDG$?

Solution. $\boxed{\textbf{Answer: } \frac{135}{2}}$

We can see that the desired area is in the shape of a trapezoid. We already know the height – it is 6 – so we just need to find the lengths of the two bases (GD and BF) and we are done with problem. Base GD is simply $\frac{30}{2} = 15$.

The harder part of the problem is finding base BF. Note that $\triangle BEF \sim \triangle DCF$ with a ratio of $\frac{2}{6} = \frac{1}{3}$. Therefore, $\frac{BF}{FC} = \frac{1}{3}$, and we have $BF = \frac{15}{2}$.

Now, we find that

$$[BFDG] = \frac{15 + \frac{15}{2}}{2} \cdot 6 = \boxed{\frac{135}{2}}$$

. □

Problem 1.16. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Solution. $\boxed{\textbf{Answer: } 20}$
Set $S = (x + y)$ and $P = xy$. Then the relationship

$$(ax^n + by^n)(x + y) = (ax^{n+1} + by^{n+1}) + (xy)(ax^{n-1} + by^{n-1})$$

can be exploited:

$$\begin{aligned} (ax^2 + by^2)(x + y) &= (ax^3 + by^3) + (xy)(ax + by) \\ (ax^3 + by^3)(x + y) &= (ax^4 + by^4) + (xy)(ax^2 + by^2) \end{aligned}$$

Therefore:

$$\begin{aligned} 7S &= 16 + 3P \\ 16S &= 42 + 7P \end{aligned}$$

Consequently, $S = -14$ and $P = -38$. Finally:

$$\begin{aligned} (ax^4 + by^4)(x + y) &= (ax^5 + by^5) + (xy)(ax^3 + by^3) \\ (42)(S) &= (ax^5 + by^5) + (P)(16) \\ (42)(-14) &= (ax^5 + by^5) + (-38)(16) \\ ax^5 + by^5 &= 20 \end{aligned}$$

□

Problem 1.17. Given that $\sin a + \sin b = \frac{1}{10}$ and $\cos a + \cos b = \frac{1}{9}$, find $\lfloor \tan^2(a + b) \rfloor$

Solution. $\boxed{\textbf{Answer: } 89}$
Square both equations to get $\sin^2 a + \sin^2 b + 2 \sin a \sin b = \frac{1}{100}$ and $\cos^2 a + \cos^2 b + 2 \cos a \cos b = \frac{1}{81}$. Add these up to get

$$\sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b + 2 \sin a \sin b + 2 \cos a \cos b = \frac{1}{100} + \frac{1}{81}$$

$$\Longrightarrow 2 + 2(\cos a \cos b + \sin a \sin b) = \frac{181}{8100} \Longrightarrow \cos a \cos b + \sin a \sin b = -\frac{16019}{16200}$$

Note that this expression is equal to $\cos(a - b)$ so we have $\cos(a - b) = -\frac{16019}{16200}$. We now take the squares of the given equations and subtract them. We have

$$\cos^2 a - \sin^2 a + \cos^2 b - \sin^2 b + 2(\cos a \cos b - \sin a \sin b) = \frac{1}{81} - \frac{1}{100}$$

Simplifying this gives us

$$\cos 2a + \cos 2b + 2 \cos(a + b) = \frac{19}{8100}$$

The cos sum to product rule says $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ so we have $\cos 2a + \cos 2b = 2\cos(a+b)\cos(a-b)$ Thus, we have

$$2\cos(a+b)\cos(a-b) + 2\cos(a+b) = \frac{19}{8100} \implies 2\cos(a+b)(\cos(a-b)+1) = \frac{19}{8100}$$

We found that $\cos(a-b) = -\frac{16019}{16200}$ so $2\cos(a+b)\cdot\frac{181}{16200} = \frac{19}{8100}$. Solving, we get that $\cos(a+b) = \frac{19}{181}$. By the Pythagorean Identity, we have that $\sin(a+b) = \frac{180}{181}$ so $\tan(a+b) = \frac{180}{19} \implies \tan^2(a+b) = \frac{32400}{361}$, which is slightly smaller than 90 so our answer is $\boxed{89}$.

□

Problem 1.18. In a triangle ABC, D is midpoint of BC . If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

Solution. $\boxed{\textbf{Answer: } 30}$

From $BD = CD$ we have $\frac{BD}{AD} = \frac{CD}{AD}$. Let $\angle BAD = x$ by sine theorem

$\frac{sinx}{sin(x+45)} = \frac{sin15}{sin30}$ or $2sinx\cos 15 = sin(x+45)$

$sin(x+15) + sin(x-15) = sin(x+45)$

$sin(x+15) = sin(x+45) - sin(x-15) = 2sin30cos(x+15) = cos(x+15)$

$x = 30.$

□

Problem 1.19. Find $a + 2b + 3c$ If

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

Solution. $\boxed{\textbf{Answer: } 2}$

This is an easy system to solve by elimination:

$a + \frac{3}{b} - 3 = a + \frac{3}{2-\frac{2}{c}} - 3 = a + \frac{3}{2-\frac{2}{1-\frac{1}{a}}} - 3 = a + \frac{3}{2-\frac{2a}{a-1}} - 3 = a - \frac{3(a-1)}{2} - 3 = \frac{-3-a}{2}$

So $a = -3$, $c = 1 + \frac{1}{3} = \frac{4}{3}$, and $b = 2 - \frac{2}{4/3} = 2 - \frac{3}{2} = \frac{1}{2}$.

$a + 2b + 3c = -3 + 2\cdot\frac{1}{2} + 3\cdot\frac{4}{3} = -3 + 1 + 4 = 2$

□

Problem 1.20. Find the minimum value of

$$f(x) = \frac{x^2+x+1}{x^2+2x+1}$$

for all x in the domain of $f(x)$

Solution. $\boxed{\textbf{Answer: } \frac{3}{4}}$

First, we can rewrite the expression as

$$1 - \frac{x}{(1+x)^2}$$

This expression is also equal to

$$1 - \frac{1}{1+x} + \frac{1}{(1+x)^2}$$

So, we can factorize that to be

$$\left(\frac{1}{1+x} - \frac{1}{2}\right)^2 + \frac{3}{4}$$

Therefore, the least value is achieved when $\frac{1}{1+x} - \frac{1}{2} = 0$, or when $x = 1$ Therefore, the least possible value is $\frac{3}{4}$. $\square$
